ELASTOTHERMO-HYDRODYNAMIC PROBLEMS IN THE THEORY OF LUBRICATION

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Formulations of stationary thermal elastohydrodynamic problems for an elastoviscous Maxwellian fluid are proposed. The basic dimensionless parameters are identified. It is shown that the problem reduces to equations with finite-dimensional-operator coefficients. A numerical solution is given for a thermal problem for a Newtonian fluid and for an isothermal problem for nonlinear-viscous fluid. The velocity, temperature, heat flux, and pressure distributions and the profile of the gap in the contact are found. The theoretical results are compared with experimental results.

1. We shall examine the stationary problem of rolling with slipping for two elastic bodies pressed together by load q. The region of the contact is filled with a lubricant, which we shall assume to be an incompressible nonlinear-viscous liquid. Let the x_1 and x_2 axes of a rectangular system of coordinates lie in the plane of the contact and let the z axis be perpendicular to this plane; i_1, i_2, k are the corresponding unit vectors. Then, when the usual assumptions for an elastohydrodynamic contact are satisfied [1], we have the following problem which describes the flow of lubricant, deformation of the bodies in contact, and propagation of heat liberated at the contact to the bodies:

$$\partial p/\partial x_i = \partial\tau_i/\partial z,\ \partial p/\partial z = 0,\ \tau_i = GC_{i3}; \tag{1.1}$$

$$2\theta' \left[(\mathbf{u}, \nabla_{12}C) - (\Omega C)^{\tau} - \Omega C\right] + C^2 - \delta' + \frac{1}{3}(I_2 - I_1)C = 0; \tag{1.2}$$

$$\det C = 1,\ I_1 = \mathrm{Sp}(C),\ I_2 = \frac{1}{2}\left(I_1^2 - \sum_{k,j=1}^{3} C_{kj}^2\right), \tag{1.3}$$

$$\theta' = \mu/G,\quad \nabla_{12} = \mathbf{i}_1\frac{\partial}{\partial x_1} + \mathbf{i}_2\frac{\partial}{\partial x_2},$$

$\Omega_{i3} = \dfrac{\partial u_i}{\partial z}$ $(i = 1, 2)$, the remaining components of Ω equal zero;

$$\rho c(\mathbf{u}, \nabla_{12}T) = k\frac{\partial^2 T}{\partial z^2} + \frac{G^2}{12\mu}(2I_1^2 + I_1I_2 - 6I_2 - 9); \tag{1.4}$$

$$\nabla_{12}\int_{-h_1}^{h_2} \mathbf{u}\,dz = 0,\ \mu = \mu_0 \exp\left[\frac{\alpha p}{1+\beta p} - \left(\delta + \frac{\varkappa p}{1+\beta p}\right)(T - T_0)\right]; \tag{1.5}$$

$$h_j = \frac{h_c}{2} + f_j + \frac{1}{E_j'}\iint_{\omega}[K_j(\xi - \mathbf{x}) - K_j(\xi)]\,p(\xi)\,d\xi; \tag{1.6}$$

$$\rho_j c_j(\mathbf{u}_j, \nabla T_j) = k_j\nabla^2 T_j,\quad \nabla = \nabla_{12} + \mathbf{k}\frac{\partial}{\partial z}; \tag{1.7}$$

$$\iint_{\omega} p(\xi)\,d\xi = q,\quad p|_{\partial\omega} = 0,\quad \left.\frac{\partial p}{\partial n}\right|_{\partial_-\omega} = 0; \tag{1.8}$$

$$u_i\left[\mathbf{x}, (-1)^j h_j\right] = U_j^i; \tag{1.9}$$

$$\left.\left(\left[\int_{-h_1}^{h_2}\mathbf{u}\,dz - \mathbf{q}_0\right], \mathbf{n}\right)\right|_{\partial_+\omega} = 0,\quad C|_{\partial_+\omega} = \delta',\quad T|_{\partial_+\omega} = T_0; \tag{1.10}$$

$$k_j\left.\frac{\partial T}{\partial n}\right|_{\gamma_{0j}} = -\lambda_j(T_j - T_0),\ k\frac{\partial T}{\partial z}\left[\mathbf{x}, (-1)^j h_j\right] = k_j\frac{\partial T_j}{\partial z}\left[\mathbf{x}, (-1)^j h_j\right], \tag{1.11}$$

$$T\left[\mathbf{x}, (-1)^j h_j\right] = T_j\left[\mathbf{x}, (-1)^j h_j\right].$$

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Here p is the pressure; τ_1, τ_2, components of the tangential stress vector; C, tensor of finite deformations; $C = (\delta' - 2\varepsilon)^{-1}$, where δ' is the unit tensor and ε is one-half the difference of the metric tensors in the deformed and undeformed states; $x = (x_1, x_2)$, $z = h_j(x_1, x_2)$, $z = f_j(x_1, x_2)$, deformed and starting geometric forms of the surfaces in contact; h_c thickness of the film at the center; u, T, ρ, c, k, μ, velocity, temperature, density, heat capacity, thermal conductivity, and viscosity (the quantities without an index refer to the lubricant and those with the indices j = 1 and 2 refer to the bodies); G, high-frequency shear modulus of the lubricant; $E'_j = E_j(1 - \nu_j^2)$, where E_j and ν_j, are the elastic moduli and Poisson's coefficients of the materials of the bodies; K_j, Green's function, which determines the normal elastic displacement of the surface of the j-th body; U_j, velocities of the surfaces neglecting the tangential displacements; λ_j, coefficients of heat transfer; μ_0, α, β, $\varkappa$, δ, constants that characterize the properties of the lubricant; ω, region of the contact, for which the input $(\partial_+\omega)$ and output $(\partial_-\omega)$ boundaries are not known beforehand and must be determined from the solution of the problem; γ_{0j}, boundaries between the bodies and the surrounding medium, which has a temperature T_0; n, outer normal; and q_0, fixed flow of lubricant at the inlet to the contact.

The calculation of the parameters of the three-dimensional elastohydrodynamic contact in the formulation given is a very complicated and laborious problem. For this reason, we shall investigate a number of cases when the inroduction of natural simplifications makes the problem more tractable and accessible to analysis and numerical solution on existing computers.

2. We shall examine a contact between cylinders made of the same material, whose axes are parallel to the direction x_2. This problem can be treated as a two-dimensional problem $(\partial/\partial x_2 = 0,\ C_{22} = 1,\ C_{12} = C_{32} = 0)$. We shall assume that the characteristic time for establishing equilibrium values of stresses in the lubricant θ' is much shorter than the residence time of a particle in the contact, and we shall neglect the convective terms in Eq. (1.2) for C_{ij}. In this case, in analogy to [1],

$$C_{11} = \sqrt{2}\frac{\sqrt{1+4\Gamma^2}}{\sqrt{1+\sqrt{1+4\Gamma^2}}},\quad C_{13} = \frac{2\Gamma}{1+\sqrt{1+4\Gamma^2}},\quad \Gamma = \theta'\frac{\partial u_1}{\partial z}$$

and the elastoviscous properties are manifested only in the nonlinear dependence of the tangential stress on the shear velocity

$$\frac{\partial u_1}{\partial z} = \frac{G}{\mu} f\left(\frac{\tau_1}{G}\right),$$

and, in addition, for the Maxwellian medium being examined $f(x) = x(1 - x^2)^{-1}$. We shall also make use of the solution of the problem of heating of cylinders with sufficiently high Peclet numbers and we shall take into account the change in temperature at the inlet due to liberation of heat in the contact [1, 2]. We shall denote $x = x_1$, $\tau = \tau_1$, $u = u_1$, and the region of the contact $a \leq x \leq d$. Let us transform to dimensionless variables: $x' = x/b$, $z' = z/h_0 - (h_2 - h_1)/2h_0$, $h' = (h_1 + h_2)/h_0$, $p' = p/p_0$, $\tau' = \tau h_0/\mu_0 U$, $u' = u/U - 1$, $\theta = \delta(T - T_0)$, where $p_0 = \sqrt{qE'/2\pi R}$ is the maximum Hertzian pressure; $b = \sqrt{8qR/\pi E'}$ is the halfwidth of the contact according to Hertz; h_0 is the thickness of the film at x = d, where $R = R_1R_2/(R_1 + R_2)$ and R_j are the radii of the cylinders; and $U = (U_1 + U_2)/2$. Green's function for cylinders replaced near the contact by half spaces equals $K(\eta) = -(4/\pi)\ln|\eta|$. Since the pressure is constant across the contact zone, we shall integrate the momentum equation. As a result, the problem assumes the following form (the primes are dropped):

$$\tau = 12\frac{H_0^2}{V}\frac{dp}{dx}z + \tau_0; \tag{2.1}$$

$$\frac{\partial u}{\partial z} = \exp\left[-\frac{Qp}{1+Np} + \Theta\left(1 + \frac{Ap}{1+Np}\right)\right]\frac{\tau}{1-(F\tau/6)^2}; \tag{2.2}$$

$$B(u+1)\left[\frac{\partial\Theta}{\partial x} + \Delta(x^2 - d^2)\frac{\partial\Theta}{\partial z}\right] = \frac{\partial^2\Theta}{\partial z^2} + H\exp\left[-\frac{Qp}{1+Np} + \Theta\left(1+\frac{Ap}{1+Np}\right)\right]\frac{\tau^2}{[1-(F\tau/6)^2]^2}; \tag{2.3}$$

$$\frac{d}{dx}\left(h + \int\limits_{-h/2}^{h/2} u\,dz\right) = 0; \tag{2.4}$$

$$H_0(h-1)=x^2-d^2+\frac{2}{\pi}\int_a^d p(\xi)\ln\frac{d-\xi}{|\xi-x|}d\xi; \tag{2.5}$$

$$p(a)=p(d)=\frac{dp}{dx}(d)=0,\ \int_a^d p(\xi)\,d\xi=\pi/2; \tag{2.6}$$

$$u(\pm h/2)=\pm\Phi,\quad h(a)+\int_{-h/2}^{h/2} udz\,|_{x=a}=M; \tag{2.7}$$

$$\Theta(\pm h/2)=\mp \mathrm{Pk}_j\int_a^x\frac{\partial\Theta}{\partial z}(\xi,\pm h/2)\frac{d\xi}{\sqrt{x-\xi}}\mp \mathrm{Bk}_j\int_a^d\frac{\partial\Theta}{\partial z}(\xi,\pm h/2)d\xi, \tag{2.8}$$

where τ_0, $=\tau(x,\ 0)$. The formulation of the problem involves the following dimensionless parameters (we shall denote the entire collection of parameters by Π_i):

$$V=\frac{3\pi\mu_0 UE'R}{q^2},\quad Q=\alpha\sqrt{\frac{qE'}{2\pi R}},$$

$$N=\frac{Q}{\alpha}\beta,\quad A=\frac{Q}{\delta\alpha}\varkappa,\quad F=\frac{6\mu_0 U}{Gh_0},\quad B=\frac{\rho cUh_0^2}{bk},$$

$$\Delta=\frac{R_1-R_2}{R_1+R_2}\frac{1}{2H_0},\quad M=\frac{q_0}{Uh_0},\quad H=\frac{\mu_0\delta U^2}{k},\quad \Phi=\frac{U_2-U_1}{2U},$$

$$\mathrm{Pk}_j=\frac{k}{k_j}\frac{b}{h_0}\mathrm{Pe}_j^{-1/2},\quad \mathrm{Pe}_j=\frac{\rho_j c_j U_j b}{k_j}\text{ — Peclet number,}$$

$$\mathrm{Bk}_j=\frac{k}{k_j}\frac{b}{2\pi h_0}\mathrm{Bi}_j^{-1},\qquad \mathrm{Bi}_j=\frac{R_j\lambda_j}{k_j}\text{ — Biot number.}$$

The last of conditions (2.6) serves to determine the dependence $H_0=H_0(\Pi_i)$, after which for given values of the physical parameters of the problem, the previously unknown quantity $h_0=H_0b^2/2R$ is found. The basic characteristics of the elastohydrodynamic contact can be represented as follows: maximum and average thickness of the lubricating film,

$$h_{\min}=h_0h'_{\min}(\Pi_i),\ h_*=h_0h'_*(\Pi_i); \tag{2.9}$$

boundaries of the contact region,

$$a=ba'(\Pi_i),\ d=bd'(\Pi_i); \tag{2.10}$$

tangential force acting on the j-th cylinder,

$$X_j=(-1)^{j+1}K_f q, \tag{2.11}$$

where the coefficient of friction K_f has the value

$$K_f=\frac{R}{b}\left[\frac{R_2-R_1}{\pi(R_2+R_1)}\int_{a'}^{d'}p'x'dx'+\frac{V}{12H_0}\int_{a'}^{d'}\tau_0'dx'\right]; \tag{2.12}$$

and heat flow from the lubricant into the cylinders and the maximum temperature of the lubricant

$$q_{Wj}=(-1)^{j+1}\frac{kb}{\delta h_0}q_j'(\Pi_i),\quad q_j'=\int_{a'}^{d'}\frac{\partial\Theta'}{\partial z'}\left[x',(-1)^j\frac{h'}{2}\right]dx', \tag{2.13}$$

$$T_{\max}=\delta^{-1}\Theta'_{\max}(\Pi_i).$$

The dimensionless functions marked with a prime on the right sides of (2.9)-(2.13) are determined from the solution of the problem (2.1)-(2.8).

3. Let the temperatures of the surfaces be fixed $\Theta[x',\ (-1)^j h/2]=\Theta_j$ and $N=0$. We introduce the variables $t=\Theta-\vartheta$, $y=z/(h/2)$, $\zeta(x,\ y)=t[x,\ (h/2)y](1+Ap)$, $\xi_0=\tau_0/2h$, $\vartheta=(\Theta_1+\Theta_2)/2$, $S=(\Theta_2-\Theta_1)/2$. In so doing, Π_c will be replaced by μ_0 in $\mu_0'=$

$\mu_0 e^{-\vartheta}$. Let us integrate (2.2) across the layer, and using the boundary conditions (2.7) we obtain the relation

$$2\Phi = e^{-Qp}h^2 \int_{-1}^{1} f\left(\xi_0, p, \frac{dp}{dx}, h, y; H_0, V, F\right) e^{\zeta} dy,$$

where $f = 3\left(\frac{H_0^2}{V}\frac{dp}{dx}y + \frac{\xi_0}{3}\right)\left[1 - F^2h^2\left(\frac{H_0^2}{V}\frac{dp}{dx}y + \frac{\xi_0}{3}\right)^2\right]^{-1}$,

which determines ξ_0 in the form of the functional

$$\xi_0 = 3\varphi\left(\zeta; h, p, \frac{dp}{dx}, H_0, V, F, \Phi\right). \tag{3.1}$$

Having found the velocity u from (2.2), we obtain the equations

$$B\left[1 - \Phi + h^2 e^{-Qp}\int_{-1}^{y} f(3\varphi)\, e^{\zeta} dy\right]\left[\frac{\partial\zeta}{\partial x} + \frac{2\Delta(x^2-d^2)}{h}\frac{\partial\zeta}{\partial y} - \frac{A\zeta}{1+Ap}\frac{dp}{dx}\right] \tag{3.2}$$

$$= \frac{\partial^2\zeta}{\partial y^2} + 4H(1+Ap)\, e^{-Qp+\zeta}h^2\left(\frac{H_0^2}{V}\frac{dp}{dx}y + \varphi\right) f(3\varphi);$$

$$\zeta(\pm 1) = \pm S(1+Ap),\ \zeta(a) = -\vartheta; \tag{3.3}$$

$$\frac{dh}{dx} - \frac{d}{dx}\left[e^{-Qp}h^2\int_{-1}^{1} f(3\varphi)\, y e^{\zeta} dy\right] = 0. \tag{3.4}$$

Equation (3.2) and the conditions (3.3) determine the temperature $\zeta(x, y)$, while Eq. (3.4), which is the analog of Reynolds' equation for a nonisothermal nonlinear-viscous liquid, together with conditions (2.6) and second condition (2.7), can be used to find the pressure p(x) and the constants a, d, and H_0. The thickness of the film h(x) is calculated from (2.5).

4. In the limiting case of a Newtonian fluid (F = 0), we have the explicit expressions

$$\xi_0 = \left(2\Phi h^{-2}e^{Qp} - 3\frac{H_0^2}{V}\frac{dp}{dx}Q_1\right)Q_0^{-1} - \frac{6\Delta(x^2-d^2)}{h}\frac{dp}{dx}; \tag{4.1}$$

$$u(x, y) = -\Phi + \frac{2\Phi}{Q_0}\int_{-1}^{y} e^{\zeta} dy + \frac{2\Phi}{Q_0Q_2 - Q_1^2}\Psi\left(Q_0\int_{-1}^{y} y e^{\zeta} dy - Q_1\int_{-1}^{y} e^{\zeta} dy\right), \tag{4.2}$$

where

$$\Psi = \left(\frac{1}{\Phi} - \frac{Q_1}{Q_0}\right) - \frac{1}{h}\left(\frac{1}{\Phi} - \frac{Q_1^*}{Q_0^*}\right),$$

$$Q_j = \int_{-1}^{1} y^j e^{\zeta} dy,\quad Q_j^* = Q_j|_{x=d}.$$

Using (4.1) and (4.2), Eqs. (3.2) and (3.4) can be written in the form

$$B\left[1 - \Phi + \frac{2\Phi}{Q_0}\int_{-1}^{y} e^{\zeta} dy + \frac{2\Phi}{Q_0Q_2 - Q_1^2}\Psi\left(Q_0\int_{-1}^{y} y e^{\zeta} dy - Q_1\int_{-1}^{y} e^{\zeta} dy\right)\right]\times \tag{4.3}$$

$$\times\left[\frac{\partial\zeta}{\partial x} + \frac{2\Delta(x^2-d^2)}{h}\frac{\partial\zeta}{\partial y} - \frac{A\zeta}{1+Ap}\frac{2}{3}\frac{V}{H_0^2}\frac{e^{(Q-A\vartheta)p}}{h^2}\frac{Q_0\Phi}{Q_0Q_2 - Q_1^2}\Psi\right] = \frac{\partial^2\zeta}{\partial y^2}$$

$$+ 4H\Phi^2(1+Ap)\frac{e^{(Q-A\vartheta)p+\zeta}}{Q_0^2}(1+\Psi)\left(\frac{y - Q_1Q_0^{-1}}{Q_2Q_0^{-1} - Q_1^2Q_0^{-2}}\right)^2;$$

$$\frac{dp}{dx} = \frac{2}{3}\frac{V}{H_0^2}\frac{e^{(Q-A\vartheta)p}}{h^2}\frac{Q_0\Phi}{Q_0Q_2 - Q_1^2}\Psi. \tag{4.4}$$

Analogous equations with A = B = 0 were obtained in [3]. For high rolling velocities, the liberation of heat due to compression of the lubricant becomes significant [4]. It can be taken into account by adding the term $\chi(1+u)\left(\frac{\zeta}{1+Ap}+\vartheta+\delta T_0\right)\frac{dp}{dx}$ on the right side of (4.3) where $\chi=\frac{\varepsilon UE'}{4\delta R}$ and $\varepsilon_0=-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)$ is the coefficient of thermal expansion.

However, for moderate rolling velocities, this factor can be neglected [4]. If, in addition, the slipping velocity (i.e., the parameter Φ), is significant, then the terms related with convective heat transfer can be dropped, i.e., we can set B = 0 in (4.3), dropping simultaneously the last condition in (3.3).

5. For convenience in carrying out the numerical solution of the system (4.3), (4.4), (3.3), and (2.6) at A = B = 0 and fixed a, we shall differentiate (4.4) with respect to x and we shall also use a different method for putting p and x into dimensionless form which eliminates the need to satisfy the last condition (2.6). For this, we refer the pressure and longitudinal coordinate, respectively, to $12\mu_0 U\sqrt{2Rh_0}/h_0^2$ and $\sqrt{2Rh_0}$ and we denote $L = 12\mu_0\alpha U\sqrt{2Rh_0}/h_0^2$, $D = 96\mu_0 UR/(\pi E' h_0^2)$. Then, we obtain

$$\frac{\partial^2\zeta}{\partial y^2}=-4H^2\Phi^2\frac{e^{Lp+\zeta}}{Q_0^2}(1+\Psi)(y-Q_1Q_0^{-1})^2(Q_2Q_0^{-1}-Q_1^2Q_0^{-2})^{-2}; \tag{5.1}$$

$$\frac{d}{dx}\left[h^3e^{-Lp}\Gamma(\zeta;p,h)\frac{dp}{dx}+\omega(\zeta;p,h)\right]=2x+D\int_a^d\frac{p(\xi)}{\xi-x}d\xi; \tag{5.2}$$

$$h=1+x^2-d^2+D\int_a^d p(\xi)\ln\frac{d-\xi}{|\xi-x|}d\xi; \tag{5.3}$$

$$p(a)=p(d)=\frac{dp}{dx}(d)=0,\quad \zeta(\pm 1)=\pm S, \tag{5.4}$$

$$\Gamma(\zeta;p,h)=\frac{3}{2}\frac{Q_0Q_2-Q_1^2}{Q_0-\Phi Q_1},\quad \omega(\zeta;p,h)=\frac{Q_0^*-\Phi Q_1^*}{Q_0-\Phi Q_1}\frac{Q_0}{Q_0^*}.$$

To solve (5.1)-(5.4) numerically, we introduce a grid $\{x_n, y_m\}$, n = 1, ..., N, m = 1,..., M, $x_1 = a$, $y_1 = -1$, $x_M = d$, $y_M = 1$ which is nonuniform in the x direction. For a pressure given in the form of a piecewise linear function, the distribution $\zeta(y)$ was found in each section $x = x_n$ by solving the boundary value problem for Eq. (5.1) numerically using the Runge–Kutta method and finding the first derivative at y = −1. The quantities $Q_j(x_n)$ determined were then used to solve the system of nonlinear difference equations approximating Eq. (5.2), using Newton's method. The value of h is given by a finite sum, replacing the integral in (5.3). The difference scheme represents the generalized schemes in [5] and has the form

$$\frac{2}{(x_{i+1}-x_{i-1})e^{Lp_i/2}}\left\{p_{i-1}\frac{h^3_{i-1/2}\Gamma_{i-1/2}}{(x_i-x_{i-1})e^{Lp_{i-1/2}}}-p_i\left[\frac{h^3_{i-1/2}\Gamma_{i-1/2}}{(x_i-x_{i-1})e^{Lp_{i-1/2}}}+\right.\right.$$
$$\left.\left.+\frac{h^3_{i+1/2}\Gamma_{i+1/2}}{(x_{i+1}-x_i)e^{Lp_{i+1/2}}}\right]+p_{i+1}\frac{h^3_{i+1/2}\Gamma_{i+1/2}}{(x_{i+1}-x_i)e^{Lp_{i+1/2}}}+(\omega_{i+1/2}-\omega_{i-1/2})\right\}$$
$$-D\sum_{\substack{k=2\\k\neq i,i+1}}^{N}\left[p_{k-1}\left(\frac{x_k-x_i}{x_k-x_{k-1}}\ln\frac{x_k-x_i}{x_{k-1}-x_i}-1\right)+\right.$$
$$\left.+p_k\left(\frac{x_i-x_{k-1}}{x_k-x_{k-1}}\ln\frac{x_k-x_i}{x_{k-1}-x_i}+1\right)\right]-D\left(p_{i+1}-p_{i-1}+p_i\ln\frac{x_{i+1}-x_i}{x_i-x_{i-1}}\right)=2x_i,$$
$$p_1=p_N=0,\quad p_{i\pm 1/2}=\frac{p_i+p_{i\pm 1}}{2},\quad x_{i\pm 1/2}=\frac{x_i+x_{i\pm 1}}{2},$$
$$h_{i\pm 1/2}=h(x_{i\pm 1/2},p_2,\ldots,p_{N-1}),\ \Gamma_{i\pm 1/2}=\Gamma(p_{i\pm 1/2},h_{i\pm 1/2}),$$
$$i=2,\ldots,N-1.$$

Determining the quantity $P=\int_a^d p\,dx$, from the results of the solution we can transform to the starting parameters

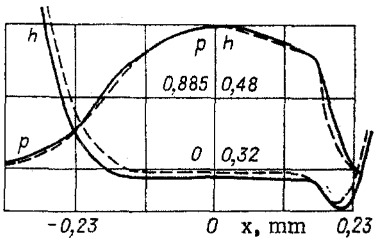

Fig. 1

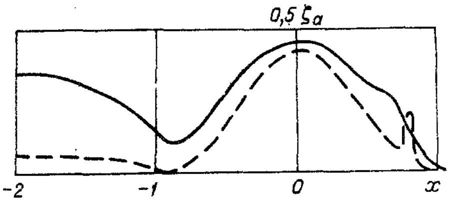

Fig. 2

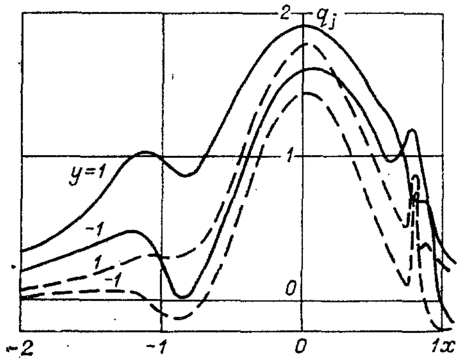

Fig. 3

$$Q=\frac{2L}{\pi}\sqrt{\frac{P}{D}},\quad V=\frac{\pi}{2}\frac{1}{DP^{2}},\quad H_{0}=\frac{1}{DP}.$$

The results of the numerical calculations are presented in Fig. 1-4. In Fig. 1, the continuous lines show graphs of the quantities $p\cdot 10^{-8}$ ($N\cdot m^{-2}$) and h (μm) from the solution of the heat problem with L = 5, D = 1.6, H = 0.5, Φ = 0.25, and S = −0.1. It is evident from a comparison with the known [5] isothermal solution for the same values of L and D (dashed line) that the influence of the thermal effects on the pressure distribution in the contact is not significant. At the same time, heat-induced thinning of the lubricating film is appreciable, especially at the inlet section. The physical parameters, in this case, have the following values: $q = 2\cdot 10^{5}$ $N\cdot m^{-2}$, $U = 10$ $m\cdot sec^{-1}$, $R = 2.3\cdot 10^{-2}$ m, $h = 0.32$ μm, $E' = 2.28\cdot 10^{11}$ $N\cdot m^{-2}$, $\mu_0 = 5.7\cdot 10^{-3}$ $N\cdot sec\cdot m^{-2}$, $\alpha = 6.6\cdot 10^{-9}$ $m^{2}\cdot N^{-2}$. Figures 2 and 3 show the distributions of the dimensionless temperature ζ_α along the center line of the film and dimensionless heat flow $q_j = (\partial\zeta/\partial z)_j$ on the upper (y = 1) and lower (y = −1) boundaries of the contact for the cases L = 5, D = 1.6, H = 0.5, Φ = 0.25, S = −0.1 (continuous line) and L = 6, D = 1.1, H = Φ = 0.1, S = −0.1 (dashed line). The corresponding functions are nonmonotonic. Sharp peaks in heat flow near the inlet zone are characteristic. The heat flow in almost the entire extent of the contact zone in the less heated body exceeds the heat flow in the more heated body, which agrees with the approximate analytic solution of the problem [1]. The quantity q_j is assumed to be positive if the heat flow is oriented from the lubricant into the body. According to Fig. 3, within a small section near the outlet zone, and for the dashed curve in the inlet section as well, heat is transferred from the more heated body into the lubricant. The temperature profiles ζ(y) and the reduced longitudinal velocity u(y) in different sections x = const are shown in Fig. 4 (curve I, x = −2.17; II, x = −1.67; III, x = −1.11; IV, x = 0; V, x = 0.57). It is interesting to note that in the inlet section there is a region where u < −1, i.e., the velocity near the axis of the contact is negative, and there is a reverse flow, in which the lubricant moves in a direction opposite to the rolling motion. Comparison of the computed values of the minimum thickness of the film with the experimental data [6] for diether oil ($\alpha = 8.2\cdot 10^{-9}$ $m^{2}\cdot N^{-1}$, $\mu_0 = 9.08\cdot 10^{-3}$ $N\cdot sec\cdot m^{-2}$) shows satisfactory agreement between the heat theory and experiment (Fig. 5). For comparison, the dashed line in Fig. 5 shows the result of a calculation of the thickness of the film using Grubin's equation. We note that in [7] the thermal elastohydrodynamic problem is solved numerically with the introduction of a number of simplifying assumptions (for example, it was assumed in [7] that the viscosity of the lubricant is constant across the layer).

6. Let us examine a two-dimensional isothermal problem for a nonlinear-viscous lubricant. In this case, the functional (3.1) gives the expression

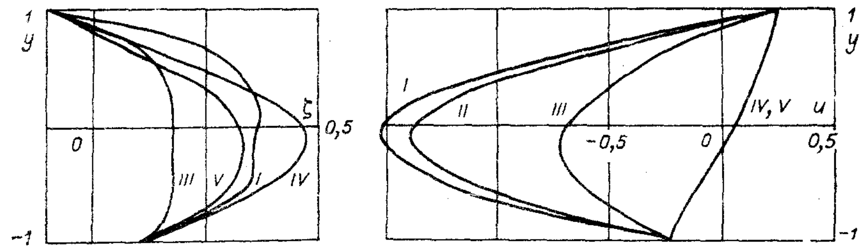

Fig. 4

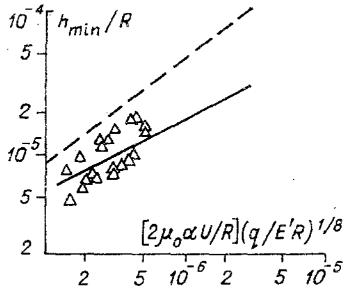

Fig. 5

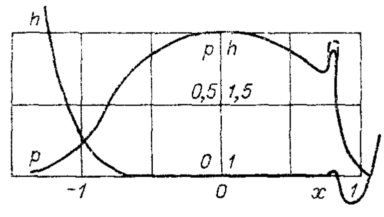

Fig. 6

$$\xi_0 = \frac{3}{2}\left[\frac{1}{Fh}\sqrt{1+\left(\frac{Fhp_x}{\operatorname{sh}\left[\frac{1}{3}SF\right]}\right)^2} - p_x \operatorname{cth}\left(\frac{1}{3}SF\right)\right], \quad S = 2\Phi F e^{Lp} p_x. \tag{6.1}$$

Relations (6.1) and (3.4) permit writing out explicitly a Reynolds equation

$$-\frac{F^2}{3}\frac{dh}{dx} = \frac{d}{dx}\left\{h e^{-Lp} p_x^{-1}\left[1 - \frac{1}{3}FS \operatorname{cth}\left(\frac{1}{3}FS\right)\right] - \right.$$
$$\left. - \frac{F^{-1}}{4} e^{Lp} p_x^{-2} \ln\left|\frac{\operatorname{ch}\left(\frac{1}{3}FS\right)\sqrt{1+(Fp_xh)^2 \operatorname{sh}^{-2}S} + \operatorname{sh} S - Fp_xh \operatorname{sh}^{-1}S}{\operatorname{ch}\left(\frac{1}{3}FS\right)\sqrt{1+(Fp_xh)^2 \operatorname{sh}^{-2}S} - \operatorname{sh} S - Fp_xh \operatorname{sh}^{-1}S}\right| + \frac{F\Phi}{3p_x}\sqrt{1+(Fp_xh)^2 \operatorname{sh}^{-2}S}\right\}. \tag{6.2}$$

in the case of pure rolling ($\Phi = 0$), we find from (6.2)

$$\frac{1}{3}F^2\frac{dh}{dx} = \frac{d}{dx}\left\{e^{-LP} p_x^{-1}\left[0.5\,(p_xF)^{-1}\ln\frac{1+Fhp_x}{1-Fhp_x} - h\right]\right\}. \tag{6.3}$$

The numerical solution of (6.3), (5.3), and (5.4) was obtained by a method based on [5]. The resulting pressure distribution and film thickness for L = 6.5 and D = 1.1 are shown in Fig. 6 for F = 0.122 (continuous line). Comparison with the solution for a Newtonian fluid (F = 0) shows that for the chosen values of the parameters, the nonlinear-viscous effects are important only for the pressure distribution in the region with the highest pressure gradients and leads to a decrease in the second peak in the pressure. Of course, the effect of nonlinear viscosity on friction is very large. In Figs. 2-4 and 6, the pressure is scaled to p_0, the longitudinal coordinate is scaled to b, and the thickness of the film is scaled to h_0.

7. We shall obtain the thermal analog of the Reynolds equation for a Newtonian lubricant at the contact of three-dimensional elastic bodies. In this case, we have (i = 1, 2)

$$\tau_i = \mu(p, T)\frac{\partial u_i}{\partial z}, \quad \rho c(\mathbf{u}, \nabla_{12}T) = k\frac{\partial^2 T}{\partial z^2} + \left(\boldsymbol{\tau}, \frac{\partial \mathbf{u}}{\partial z}\right),$$

$$h_j(x_1, x_2) = \frac{h_c}{2} + f_j + \frac{1}{\pi E_j'}\iint_\omega\left[\frac{1}{\sqrt{(x_1-\xi)^2+(x_2-\eta)^2}} - \frac{1}{\sqrt{\xi^2+\eta^2}}\right]p(\xi, \eta)\,d\xi d\eta.$$

Since the components of the tangential stress factor equal

$$\tau_i = \frac{\partial p}{\partial x_i} z + \tau_{0i},$$

the velocity components can be calculated from the equations

$$u_i = U_1^i + \int_{h_1}^{z} \left(\frac{\partial p}{\partial x_i} z_1 + \tau_{0i} \right) \frac{dz_1}{\mu(p, T)}. \tag{7.1}$$

Satisfying the sticking condition at $z = h_2$, we obtain the expressions

$$U_2^i - U_1^i = \int_{-h_1}^{h_2} \left(\frac{\partial p}{\partial x_i} z + \tau_{0i} \right) \frac{dz}{\mu(p, T)},$$

from which the quantities τ_{0i} are determined in the form of the functionals

$$\tau_{0i} = \varphi_i\left(T, p, \frac{\partial p}{\partial x_i}, h_1, h_2\right). \tag{7.2}$$

Substituting (7.1) and (7.2) into the equation of continuity leads to a Reynolds equation for the pressure

$$\sum_{i=1}^{2} \left[U_2^i \frac{\partial h_2}{\partial x_i} + U_1^i \frac{\partial h_1}{\partial x_i} - \frac{\partial}{\partial x_i} \int_{-h_1}^{h_2} \left(\frac{\partial p}{\partial x_i} z + \varphi_i \right) \frac{dz}{\mu(p, T)} \right] = 0.$$

The temperature distribution must be determined from the equation

$$\rho c \sum_{i=1}^{2} \left\{ \left[U_1^i + \int_{-h_1}^{z} \left(\frac{\partial p}{\partial x_i} z_1 + \varphi_i \right) \frac{dz_1}{\mu(p, T)} \right] \frac{\partial T}{\partial x_i} - \frac{1}{\mu(p, T)} \left(\frac{\partial p}{\partial x_i} z + \varphi_i \right)^2 \right\} = k \frac{\partial^2 T}{\partial z^2}.$$

All relations at a given point are written in dimensional variables. When transforming to dimensionless quantities, it is necessary to use the maximum pressure p_0 at the contact and the characteristic sizes a and b of the region of contact ω, which in each specific case are determined from the solution of the corresponding contact problem of the theory of elasticity. For example, for a Hertzian contact, ω is an ellipse with semiaxes a and b and the maximum pressure is $p_0 = 3q/2\pi ab$.

The thermal analog of the Reynolds equation obtained above is applicable to contact interactions in ball and roller bearings, slipping bearings, gears, and also contact of a piston ring with the cylinder in an engine.

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AN EXPERIMENTAL AND THEORETICAL INVESTIGATION OF THE STABILITY OF A LINEAR VORTEX WITH A DEFORMED CORE

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UDC 532.527+532.516

Experiments and a mathematical model of the instability of a linear vortex subject to deformation such that the streamlines are nearly ellipses with small eccentricities are described in the report. The tests were carried out with a draining-vortex type of flow in a cylindrical vessel with an elliptical cross section. The wavelengths and rotation rates of unstable modes were measured. An analytical model of the instability is proposed, based on linear theory using perturbation theory relative to the smallness of the deformation. According to this model, the mechanism of the observed instability is analogous to the instability of a wave of finite amplitude in a three-wave interaction [1, 2]. The predictions of the model explain the experimental results fairly well.

These tests can be considered as a generalization of the experiments of [3, 4] on the stability of initially rigid-body rotation inside an elliptical cylinder after it is stopped. The proposed theory of the phenomenon can also be considered as a generalization of that of [5], in which the question of the stability of a linear vortex in an unbouded fluid was investigated. The core of the vortex was assumed to be subject to deformation such that the shape of its cross section is close to an ellipse with a small eccentricity. The method of solution of [5] is used below. A theory for the above-mentioned experiments was constructed in [3, 4] on the basis of the assumption that the vorticity is constant. In contrast to the present work, Galerkin's method was used. Thus, the results of [3-5] comprise two different limiting cases of the problem under consideration.

1. Let us consider the plane stationary flow of an ideal fluid, consisting of a linear vortex with a core of constant vorticity, which is inside a cylindrical vessel. Outside the core the flow is potential. The shapes of streamlines and of the boundary of the normal cross section of the vessel differ little from circles. The quantity $\varepsilon \ll 1$ serves as the measure of this difference. In the cylindrical coordinate system (r, θ, z) we assign the flow in the form of expansions in powers of the parameter ε,

$$\left.\begin{aligned} U(r,\theta) &= -\varepsilon r\sin 2\theta + O(\varepsilon^2),\\ V(r,\theta) &= r - \varepsilon r\cos 2\theta + O(\varepsilon^2),\\ P(r,\theta) &= (1/2)r^2 + O(\varepsilon^2), \end{aligned}\right\} 0 < r \leqslant R_1(\theta), \tag{1.1}$$
$$\Phi(r,\ \theta) = \theta - (\varepsilon/4)(r^2 - r^{-2})\sin 2\theta + O(\varepsilon^2),\ R_1(\theta) \leqslant r \leqslant R_2(\theta),$$

where $R_1(\theta)$ and $R_2(\theta)$ give the boundaries of the vorticity core and the vessel,

$$\begin{aligned} R_1(\theta) &= 1 + (\varepsilon/2)\cos 2\theta + O(\varepsilon^2),\\ R_2(\theta) &= b[1 + (\varepsilon/4)B\cos 2\theta + O(\varepsilon^2)]. \end{aligned} \tag{1.2}$$

The radial and angular components of the velocity and pressure inside the vorticity core are designated as U, V, and P; Φ is the velocity potential outside this core; b is a constant equal to the vessel raidus in the zeroth approximation ($b \geqslant 1$); $B = b^2 + b^{-2}$. A system of units is used in which the vorticity in the core equals zero while the unperturbed radius of the core equals one. The first two terms of the expansion of the exact solution obtained in [6] (cited in [5]) are written out explicitly in (1.1) and (1.2). These expansions can also be obtained by a direct solution of the equations of motion by successive approximations satisfying the conditions of nonpenetration at $r = R_2(\theta)$.

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